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Time reversal symmetry breaking and exceptional points

W D Heiss

Institute of Theoretical Physics and Department of Physics, University of Stellenbosch,
7602 Matieland, South Africa

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Abstract

Exceptional points for a two-dimensional matrix Hamiltonian that breaks time reversal symmetry are revisited. Particular attention is paid to C-points where the wavefunction exhibits circular polarization. They are different from exceptional points if and when time reversal symmetry is broken, or in crystal optics, when the crystal is optically active.

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It has been discussed in [1] that there are, for a general crystal, different types of singularities with different physical manifestations. In particular, when the crystal is optically active, the chirality changes the symmetry of the dielectric tensor. As a result, the wavefunctions at the exceptional points (EP) no longer give rise to a circularly polarized state [2]—that is to the form $\{\pm i, 1\}$ —whereas circular polarization does occur at points different from the EPs, called C-points in [1]. In [3] a calculation explains why the C-points are statistically expected to be close to the EPs. In view of the close formal and physical similarity of chiral crystals and time reversal symmetry breaking Hamiltonians we here address the same problem using the notation of a previous paper [4]. The present paper refrains from repeating a statistical calculation, yet it is expected to further elucidate the fascinating physics of this general topic, and the different parametrization appears appropriate for an experiment pending [5].

We begin with the familiar setting $H_0 + \lambda H_1$ with non-commuting Hermitian 2×2 matrices H_0 and H_1 . By analytic continuation in λ of the spectrum $E_k(\lambda)$ and the state vectors $|\psi(\lambda)_k\rangle$ the EPs are found as square-root singularities where the two energy levels and the state vectors coalesce. If time reversal symmetry is broken by H_0 and/or H_1 a richer structure emerges [4]; we briefly recapitulate the results of [4] and expand where appropriate for the present paper.

Using the form

$$\begin{aligned} H &= H_0 + \lambda H_1 & \text{where} \\ H_k &= U(\phi_k, \tau_k) D_k U^\dagger(\phi_k, \tau_k), & k = 0, 1 \end{aligned} \quad (1)$$

with

$$U(\phi, \tau) = \begin{pmatrix} \cos \phi & -\sin \phi e^{i\tau} \\ \sin \phi e^{-i\tau} & \cos \phi \end{pmatrix} \quad (2)$$

and D_k being real diagonal matrices but not multiples of the unit matrix, the EPs are found at

$$\lambda_{\text{EP}} = q \exp(\pm 2i\beta) \quad (3)$$

with

$$\cos \beta = \sqrt{(\cos \phi_0 \cos \phi_1)^2 + (\sin \phi_0 \sin \phi_1)^2 + 2 \cos \phi_0 \cos \phi_1 \sin \phi_0 \sin \phi_1 \cos(\tau_0 - \tau_1)},$$

$$q = -\frac{D_0(1) - D_0(2)}{D_1(1) - D_1(2)}$$

and $D_k(i)$ denoting the entries of D_k .

Note that, for the special case $\tau_0 = \tau_1$ (denoted by τ below), it is $\beta = \phi_0 - \phi_1$. Only for this particular case, the eigenfunctions at the EP are given by (up to a factor)

$$|\psi_{\text{EP}}\rangle = \begin{pmatrix} \pm i e^{i\tau} \\ 1 \end{pmatrix}, \quad (4)$$

and they retain their form under any basis transformation of the form $U(\phi, \tau)$. In other words, the particular choice of the basis that would render H_0 diagonal does not alter the form of (4). Note also, that in this special case, the time reversal operator \mathcal{T} is simply the complex conjugation K . This simple form has to be generalized into a general anti-unitary operator when deviating from the special case $\tau_0 = \tau_1$.

In fact, we first observe that for $\tau_0 \neq \tau_1$, the state vector $|\psi_{\text{EP}}\rangle$ does depend on the basis chosen. Using, for instance the basis in which H_0 is diagonal, we consider

$$\tilde{H} = D_0 + \lambda U^\dagger(\phi_0, \tau_0) H_1 U(\phi_0, \tau_0) \quad (5)$$

which yields

$$|\psi_{\text{EP}}^{\tilde{H}}\rangle = \begin{pmatrix} \pm i e^{i\xi} \\ 1 \end{pmatrix} \quad (6)$$

where ξ is a function of ϕ_0 , τ_0 , ϕ_1 and τ_1 . Its explicit form is of little interest here, it is given in [4]. It can never be zero except when $\tau_0 = 0 = \tau_1$. Secondly we note that the time reversal operator is now given by the more complicated form $\mathcal{T} = U^\dagger(\phi_0, \tau_0) K U(\phi_0, \tau_0)$ being of the general anti-unitary form $\mathcal{T} = \tilde{U} K$ with the unitary operator $\tilde{U} = U^\dagger(\phi_0, \tau_0) U^*(\phi_0, \tau_0)$ where the asterisk denotes complex conjugation. Note that \tilde{U} is basis dependent. We conclude that, as soon as time reversal invariance is broken, the state vector at the EP can no longer have the simple form of a pure circular polarization. Transforming back to the original basis, we would consider

$$|\psi_{\text{EP}}^H\rangle = U(\phi_0, \tau_0) |\psi_{\text{EP}}^{\tilde{H}}\rangle$$

being a more complicated expression given in [4]. For $\tau_0 \neq 0 \neq \tau_1$, the ratio of the two components can be any complex number (except $\pm i$) depending on the parameters ϕ_0 , τ_0 , ϕ_1 and τ_1 . Particular choices yield special cases such as a pure linear polarization, e.g. $\{\pm 1, 1\}$ or even $\{0, 1\}$ and $\{1, 0\}$.

Recall that at the EP there is only one eigenvector and that the norm $\langle \tilde{\psi}_{\text{EP}} | \psi_{\text{EP}} \rangle$ vanishes (we denote by $\langle \tilde{\psi}_{\text{EP}} |$ the left eigenvector of H , or $|\tilde{\psi}_{\text{EP}}\rangle$ is the eigenvector of $H_0 + \lambda_{\text{EP}}^* H_1$).

We now turn to the C-points [1, 3] where one eigenfunction is of the form $\{\pm i, 1\}$. As the requirement puts a condition on the state vector, it is expected that the values λ_C being sought are basis dependent. In the basis employed in (1) we find

$$\lambda_C = q \frac{\cos 2\phi_0 \pm i \cos \tau_0 \sin 2\phi_0}{\cos 2\phi_1 \pm i \cos \tau_1 \sin 2\phi_1} \quad (7)$$

with same q as in (3).

Of course, for $\tau_0 = \tau_1 = 0$, that is when both, H_0 and H_1 are real, we retrieve (3) with $\beta = \phi_0 - \phi_1$ and the eigenvector has the form $\{\pm i, 1\}$. In other words, the C-points and the EPs coincide only if $\tau_0 = \tau_1 = 0$. These two significant points—the EPs and the C-points—are always different as soon as either τ_0 or τ_1 or both are switched on, that is as soon as time reversal invariance is broken. Also, the lowest order deviation of the C-point from the EP is quadratic in τ_0 and τ_1 as is found in [3]. An interesting point is that λ_C can occur at any real value (thus giving rise to real energies) for the special case $\tau_0 = \tau_1 = \pi/2$. This is in stark contrast to an exceptional point occurring always only at complex values for Hermitian H_0 and H_1 . If λ_C is real, both eigenstates are circularly polarized, the one being $\{+i, 1\}$ and the other $\{-i, 1\}$. Note that for the general case, when both λ_C are complex, only one is accessible in the laboratory as only one is associated with an energy having a negative imaginary part; whether the corresponding state vector is $\{+i, 1\}$ or $\{-i, 1\}$ depends on the parameters. If these parameters can be controlled in the lab, either choice can be achieved.

For completeness we discuss the basis used in (5). To emphasize the basis dependence and thus the difference to the result in (7) we use the symbol λ_c instead of λ_C . The general expression is somewhat clumsy, its inverse reads

$$\begin{aligned} (\lambda_c)^{-1} = \frac{-1}{q} \left\{ \left(\cos \frac{\tau_0 - \tau_1}{2} \right)^2 (\cos(2\phi_0 - 2\phi_1) \mp i \cos \tau_0 \sin(2\phi_0 - 2\phi_1)) \right. \\ \left. + \left(\sin \frac{\tau_0 - \tau_1}{2} \right)^2 (\cos(2\phi_0 + 2\phi_1) \mp i \cos \tau_0 \sin(2\phi_0 + 2\phi_1)) \right. \\ \left. \pm i \sin(\tau_0 - \tau_1) \sin \tau_0 \sin(2\phi_1) \right\} \quad (8) \end{aligned}$$

but the special case $\tau_0 = \tau_1$ simplifies to

$$\lambda_c = \frac{-q}{\cos 2(\phi_0 - \phi_1) \mp i \cos \tau \sin 2(\phi_0 - \phi_1)}. \quad (9)$$

The whole discussion of the previous paragraph applies *verbatim*. It is interesting to note that expansion of (8) in powers of τ_0 and τ_1 also yields odd powers in τ_0 or τ_1 ; however, the sum of the powers is always even.

To summarize: the C-points are always different from the EPs if (and only if) time reversal symmetry is broken, that is when either H_0 or H_1 is non-real, i.e. when either τ_0 or τ_1 is nonzero. In contrast to the EPs the C-points can occur on the real axis. While the EPs are basis independent, the C-points do depend on the basis considered (λ_c is different from λ_C). In an experiment such dependence can mean that one quantity (say the electric field vector) is circularly polarized at one parameter point while another (say the magnetic field vector) is circularly polarized at another point.

We conclude this paper by emphasizing that, while the C-points are of special physical interest, they do not constitute a singularity of the operator as the EPs do. In fact at any point in the λ -plane, *except at the EPs*, we have a complete bi-orthogonal system in that

$$\sum_k \frac{|\psi(\lambda)_k\rangle \langle \tilde{\psi}(\lambda)_k|}{\langle \tilde{\psi}(\lambda)_k | \psi(\lambda)_k \rangle} = 1. \quad (10)$$

This breaks down only at the EP where $|\psi(\lambda_{EP})_1\rangle$ becomes aligned to $|\psi(\lambda_{EP})_2\rangle$, and their respective scalar products vanish.

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